A SOLUTION OF DIRECT AND CONVERSE TWO-DIMENSIONAL PROBLEMS OF NONSTATIONARY HEAT CONDUCTION WITH VARIABLE BOUNDARY CONDITIONS USING THE METHOD OF TRANSFER FUNCTIONS

B. I. Strikitsa

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A procedure is developed for determining transfer functions and by employing them to solve direct and converse two-dimensional problems of nonstationary heat conduction with variable boundary conditions for a hollow cylinder.

The proposed procedure for solving direct or converse problems of nonstationary heat conduction for a hollow cylinder is a continuation of the procedure developed in [1]. In the above cited article the system ot algebraic equations (3) is analogous to the equation

$$\vartheta(r,\ \Theta,\ 0)=\int_0^{2\pi}\vartheta_{1st}(r,\ \Theta,\ \varphi)\,q_{st}(R_2,\ \varphi)\,d\varphi,$$
$$R_1<r<R_2;\quad 0<\Theta<2\pi;\quad 0<\varphi<2\pi, \tag{1}$$

where $\vartheta_{1st}(r, \Theta, \varphi)$ is the unit response function representing the response of the cylinder in the form of temperature distribution to a consecutive incidence at each point of the outer surface of the cylinder $r = R_2$ of a heat flux density $q(R_2, \Theta) = 1(\Theta)$ if on the inner cylinder surface $r = R_1$ the temperature $\vartheta(R_1, \Theta, 0) = 0$ is maintained. The temperature of the inner cylinder surface t_0 is adopted as the origin of the scale and the notation is introduced $\vartheta = t - t_0$.

The solution of direct and converse problems of nonstationary heat conduction for a hollow cylinder with boundary conditions of the second kind for $r = R_2$ and of the first kind for $r = R_1$ is represented in the form of an equation similar to Eq. (1). The direct problem is formulated as follows.

The initial condition is given in the form of a stationary temperature distribution which can be expressed by Eq. (1). At the instant $\tau = 0$ the outer surface of the cylinder begins to be subjected to the heat flux of density $q(R_2, \Theta, \tau)$ which is varying in time as well as on the circumference; the temperature of the inner surface varies according to a given law $\eta(\Theta, \tau)$. It is required to determine the temperature distribution in the cylinder at any time instant.

One has:

$$\frac{\partial\vartheta(r,\ \Theta,\ \tau)}{\partial\tau}=a\left[\frac{\partial^2\vartheta(r,\ \Theta,\ \tau)}{\partial r^2}+\frac{1}{r}\frac{\partial\vartheta(r,\ \Theta,\ \tau)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\vartheta(r,\ \Theta,\ \tau)}{\partial\Theta^2}\right],\quad(\tau>0), \tag{2}$$

$$-\frac{\partial\vartheta(R_2,\ \Theta,\ \tau)}{\partial n}+\frac{q(R_2,\ \Theta,\ \tau)}{\lambda}=0, \tag{3}$$

$$\vartheta(R_1,\ \Theta,\ \tau)=\eta(\Theta,\ \tau) \tag{4}$$

together with Eq. (1).

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Five auxiliary problems are considered for the differential equation (2) with boundary conditions which are parts of the conditions (1), (3), and (4). The first problem is given by Eqs. (1) and (2) together with the conditions

$$-\frac{\partial\vartheta(R_2,\ \Theta,\ \tau)}{\partial n}=0, \tag{5}$$

$$\vartheta(R_1,\ \Theta,\ \tau)=0; \tag{6}$$

the second problem is given by (2) and (6) and the conditions:

$$\vartheta(r,\ \Theta,\ 0)=0, \tag{7}$$

$$-\frac{\partial\vartheta(R_2,\ \Theta,\ \tau)}{\partial n}+\frac{q_{st}(R_2,\ \Theta)}{\lambda}=0; \tag{8}$$

the third problem is given by (2), (5), and (7) and the condition

$$\vartheta(R_1,\ \Theta,\ \tau)=\vartheta(R_2,\ \Theta,\ 0); \tag{9}$$

the fourth problem is given by (2), (6), and (7) and the condition

$$-\frac{\partial\vartheta(R_2,\Theta,\ \tau)}{\partial n}+\frac{q(R_2,\ \Theta,\ \tau)-q_{st}(R_2,\ \Theta)}{\lambda}=0; \tag{10}$$

the fifth problem is given by (2), (5), and (7) and the condition

$$\vartheta(R_1,\ \Theta,\ \tau)=\eta(\Theta,\ \tau)-\vartheta(R_2,\ \Theta,\ 0). \tag{11}$$

In accordance with the superposition principle which is valid for linear problems the solution of the problem (1)-(4) can be represented as a sum of the solutions of five auxiliary problems. Let the solution of the problem (1), (2), (5), and (6) with $q_{st}(R_2,\ \Theta)=1(\Theta)$ be a function $\vartheta_{\mathrm{I}}^{p\,\mathrm{II}}(r,\ \Theta,\ \varphi,\ \tau)$, the solution of the problem (2), (6), (7), and (8) for $q_{st}(R_2,\ \Theta)=1(\Theta)$ be a function, $\vartheta_1^{\mathrm{II}\,2}(r,\ \Theta,\ \varphi,\ \tau)$, and of the problem (2), (5),(7), and (9) for $\vartheta(R_2,\ \Theta,\ 0)=1(\Theta)$ a function $\vartheta_1^{\mathrm{II}\,1}(r,\ \Theta,\ \varphi,\ \tau)$. Then using the Duhamel integral the solution of the original problem (1)-(4) can be written as

$$\begin{aligned}\vartheta(r,\ \Theta,\ \tau)&=\int_0^{2\pi}\vartheta_1^{p\,\mathrm{II}}(r,\ \Theta,\ \varphi,\ \tau)\,q_{st}(R_2,\ \varphi)\,d\varphi\\
&+\int_0^{2\pi}\vartheta_1^{\mathrm{II}\,2}(r,\ \Theta,\ \varphi,\ \tau)\,q_{st}(R_2,\ \varphi)\,d\varphi+\int_0^{2\pi}\vartheta_1^{\mathrm{II}\,1}(r,\ \Theta,\ \varphi,\tau)\,\vartheta(R_2,\ \varphi,\ 0)\,d\varphi\\
&+\int_0^{\tau}\int_0^{2\pi}[q(R_2,\ \varphi,\ \xi)-q_{\text{ст}}(R_2,\ \varphi)]\frac{\partial}{\partial\tau}\vartheta_1^{\mathrm{II}\,2}(r,\ \Theta,\ \varphi,\ \tau-\xi)\,d\varphi\,d\xi\\
&+\int_0^{\tau}\int_0^{2\pi}[\eta(\varphi,\ \xi)-\vartheta(R_2,\ \varphi,\ 0)]\frac{\partial}{\partial\tau}\vartheta_1^{\mathrm{II}\,1}(r,\ \Theta,\ \varphi,\ \tau-\xi)\,d\varphi d\xi.\end{aligned} \tag{12}$$

By using the equality

$$\int_0^{2\pi}\vartheta_1^{\mathrm{II}\,1(2)}(r,\ \Theta,\ \varphi,\ \tau)\,q_{st}(R_2,\ \varphi)\,d\varphi=\int_0^{\tau}\int_0^{2\pi}q_{st}(R_2,\ \varphi)\frac{\partial}{\partial\tau}\vartheta_1^{\mathrm{II}\,1(2)}(r,\ \Theta,\ \varphi,\ \tau-\xi)\,d\varphi d\xi, \tag{13}$$

Eq. (12) can be replaced by

$$\begin{aligned}\vartheta(r,\ \Theta,\ \tau)&=\int_0^{2\pi}\vartheta_1^{p\mathrm{II}}(r,\ \Theta,\ \varphi,\ \tau)\,q_{st}(R_2,\ \varphi)\,d\varphi+\int_0^{\tau}\int_0^{2\pi}q(R_2,\ \varphi,\ \xi)\\
&\times\frac{\partial}{\partial\tau}\vartheta_1^{\mathrm{II}\,2}(r,\ \Theta,\ \varphi,\ \tau-\xi)\,d\varphi d\xi+\int_0^{\tau}\int_0^{2\pi}\eta(\varphi,\ \xi)\frac{\partial}{\partial\tau}\vartheta_1^{\mathrm{II}\,1}(r,\ \Theta,\ \varphi,\ \tau-\xi)\,d\varphi d\xi.\end{aligned} \tag{14}$$

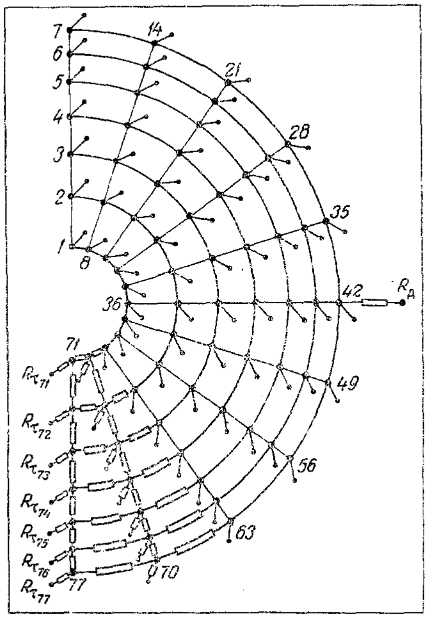

Fig. 1. Two-dimensional grid model of a half of a hollow cylinder.

The function $\vartheta_1^{p\,\mathrm{II}}(r, \Theta, \varphi, \tau)$ is called by us a relaxation unit-response function, $\vartheta_1^{\mathrm{II}\,1}(r, \Theta, \varphi, \tau)$ is called the first fundamental unit-response function and $\vartheta_1^{\mathrm{II}\,2}(r, \Theta, \varphi, \tau)$ the second fundamental unit-response function. A converse problem can also be solved by employing these functions. The converse problem can be formulated as follows.

Temperature distributions in the cylinder are given for $r = R_1$ and also on some other fixed radius for any time: $\eta(\Theta, \tau)$ and $\vartheta(r_\Phi, \Theta, \tau)$. The initial condition is represented by Eq. (1). It is required to determine the density distribution of the heat flow for $r = R_2$.

The given temperature distributions are substituted in Eq. (14); then the solution of the converse problem can be written as

$$\vartheta(r_\Phi, \Theta, \tau) = \int_0^{2\pi} \vartheta_1^{p\,\mathrm{II}}(r_\Phi, \Theta, \varphi, \tau)\, q_{st}(R_2, \varphi)\, d\varphi + \int_0^{\tau}\int_0^{2\pi} \times q(R_2, \varphi, \xi) \frac{\partial}{\partial\tau} \vartheta_1^{\mathrm{II}\,2}(r_\Phi, \Theta, \varphi, \tau-\xi)\, d\varphi d\xi + \int_0^{\tau}\int_0^{2\pi} \eta(\varphi, \xi) \frac{\partial}{\partial\tau} \vartheta_1^{\mathrm{II}\,1}(r_\Phi, \Theta, \varphi, \tau-\xi)\, d\varphi d\xi. \tag{15}$$

Equation (15) represents a mixed Fredholm–Volterra integral equation of the first kind with respect to the unknown $q(R_2, \Theta, \tau)$.

The unit-response functions were determined with the aid of an electric model by using the integrator EGDA. These functions were found for a cylinder one of whose diameter planes is the symmetry plane for the temperature field. A two-dimensional network model was therefore constructed of a half of a hollow cylinder, as shown in Fig. 1. The method employed to determine these functions is identical with that for determining the functions Φ in [1]. A potential equal to 100% is applied to the additional resistance R_{add}. The resistance R_{add} is connected consecutively to all nodes on the outer circumference of the model. The potential equal to 0% is applied to all nodes of the inner circumference. The resistances R_τ are disconnected. The potentials $U_{st}(r_k, \Theta_i, \varphi_j)$ are now measured on all nodes of the model. In the above we have $k = 1, 2, 3, \ldots, 7$; $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, m = n$. The value of the current I_B applied consecutively to the nodes of the model outer circumference is determined after the voltage drop on the resistance R_{add}. The values of the stationary unit response function are found as the ratio of the quantities $U_{st}(r_k, \Theta_i, \varphi_j)$ and $I_B(\Theta_i)$

$$\Phi_{st}^{\mathrm{II}}(r_k, \Theta_i, \varphi_j) = U_{st}(r_k, \Theta_i, \varphi_j)/I_B(\Theta_i). \tag{16}$$

We shall now find the relaxation unit-response function. To this end potentials are applied to the nodes of the model inner circumference the potentials remaining equal to 0% in the course of the solution. In accordance with [2] to the time resistances R_τ the potentials $U_{st}(r_k, \Theta_i, \varphi_j)$ are applied at the first time instant. No inputs are sent through R_{add} to the model. The potentials $U_{p1}^{\mathrm{II}}(r_k, \Theta_i, \varphi_j, \tau_N)$ are now measured on all nodes. In the above $N = 1, 2, 3, \ldots, \infty$. The value of the relaxation unit-response function is found as the ratio of the quantities $U_{p1}^{\mathrm{II}}(r_k, \Theta_i, \varphi_j, \tau_N)$ and $I_B(\Theta_i)$, the latter being found when determining $\Phi_{st}^{\mathrm{II}}(r_k, \Theta_i, \varphi_j)$:

$$\Phi_{p1}^{\mathrm{II}}(r_k, \Theta_i, \varphi_j, \tau_N) = U_{p1}^{\mathrm{II}}(r_k, \Theta_i, \varphi_j, \tau_N)/I_B(\Theta_i). \tag{17}$$

The first fundamental unit-response function is now determined. To this end the potential U_B/c is applied consecutively to the nodes of the model inner circumference and the potential U_B/c is kept constant in the course of the solution. In the above c is a parameter. On R_τ the potentials equal to 0% are applied at the first instant. No disturbances are sent to the model through R_{add}. The potentials $U_1^{\mathrm{II}\,1}(r_k, \Theta, \varphi_j, \tau_N)$ are now measured at the nodes. The values of the first fundamental unit-response function are found as the ratio of the quantities $U^{\mathrm{II}\,1}(r_k, \Theta_i, \varphi_j, \tau_N)$ and $U_B(\Theta_i)/c$,

$$\Phi_1^{\mathrm{II}\,1}(r_k, \Theta_i, \varphi_j, \tau_N) = U_1^{\mathrm{II}\,1}(r_k, \Theta_i, \varphi_j, \tau_N)\, c/U_B(\Theta_i). \tag{18}$$

The second fundamental unit-response function is now found. To the nodes of the model outer circumference a current of magnitude I_B/c is applied through R_{add} and remains constant during the course of the solution. Potentials are applied to the model inner circumference and are maintained in the course of the solution as equal to 0%. To R_τ potentials equal to 0% are applied at the first instant. The potentials $U_1^{II\,2}(r_k, \Theta_i, \varphi_j, \tau_N)$ are measured at the nodes. The values of the second fundamental unit-response function are found as the ratio of $U_1^{II\,2}(r_k, \Theta_i, \varphi_j, \tau_N)$ and $I_B(\Theta_i)/c$,

$$\Phi_1^{II\;2}(r_k, \Theta_i, \varphi_j, \tau_N) = U_1^{II\,2}(r_k, \Theta_i, \varphi_j, \tau_N)\,c/I_B(\Theta_i). \tag{19}$$

The parameter c is introduced if one finds that for some instant one should change the potential U_B or the current I_B c times. Then the potentials $U_1^{II1}(r_k, \Theta_i, \varphi_j, \tau_N)$ and $U_1^{II\,2}(r_k, \Theta_i, \varphi_j, \tau_N)$ obtained at the preceding instant and supplied to R_τ must be changed c times, correspondingly.

Relations will be established between the unit-response functions expressed in electrical units and the unit-responsefunctions expressed in thermal units. Following [1] it is easily shown that these relations are as follows:

$$\vartheta_1^{p\,II}(r_k, \Theta_i, \varphi_j, \tau_N)\,l = \Phi_{p1}^{II}(r_k, \Theta_i, \varphi_j, \tau_N), \tag{20}$$

$$\vartheta_1^{II\,2}(r_k, \Theta_i, \varphi_j, \tau_N)\,l = \Phi_1^{II\,2}(r_k, \Theta_i, \varphi_j, \tau_N), \tag{21}$$

$$\vartheta_1^{II\,1}(r_k, \Theta_i, \varphi_j, \tau_N) = \Phi_1^{II\,1}(r_k, \Theta_i, \varphi_j, \tau_N). \tag{22}$$

In the above

$$l = C_R m^*/B_e, \tag{23}$$

where C_R is the resistance scale; m^* is the scale of the model; B_e is the length of the portion of the model outer circumference through which the input is applied to the model when unit-response functions are determined. By using the relations (20), (21), and (22) and following [3] Eqs. (14) and (15) are represented in the form of a system of algebraic equations. Thus the solution of the direct problem of nonstationary heat conduction with boundary conditions of the second kind for $r = R_2$ and for the first kind for $r = R_1$ takes the form of an analog of Eq. (14):

$$\vartheta(r_k, \Theta_i, \tau_N) = \frac{1}{l}\sum_{j=1}^{n}\Phi_{p1}^{II}(r_k, \Theta_i, \varphi_j, \tau_N)\,q_{st}(R_2, \varphi_j)$$
$$+\frac{1}{l}\sum_{z=1}^{N}\sum_{j=1}^{n}[\Phi_1^{II\,2}(r_k, \Theta_i, \varphi_j, \tau_{N-z+1}) - \Phi_1^{II\,2}(r_k, \Theta_i, \varphi_j, \tau_{N-z})]\,q(R_2, \varphi_j, \tau_z)$$
$$+\sum_{z=1}^{N}\sum_{j=1}^{n}[\Phi_1^{II\,1}(r_k, \Theta_i, \varphi_j, \tau_{N-z+1}) - \Phi_1^{II\,1}(r_k, \Theta_i, \varphi_j, \tau_{N-z})]\,\eta(\varphi_i, \tau_z). \tag{24}$$

The solution of the converse problem is analogous to Eq. (15). The obtained equation differs from Eq. (24) in that the sought function $q(R_2, \Theta_i, \tau_N)$ appears within the summation signs. It can easily be shown that this equation can be given by

$$\sum_{j=1}^{n}\Phi_1^{II\,2}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_1)\,q(R_2, \varphi_j, \tau_N) = l\vartheta(r_{k\Phi}, \Theta_i, \tau_N)$$
$$-\sum_{j=1}^{n}\Phi_{p1}^{II}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_N)\,q_{st}(R_2, \varphi_j) - \sum_{z=1}^{N-1}\sum_{j=1}^{n}[\Phi_1^{II\,2}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_{N-z+1})$$
$$-\Phi_1^{II\,2}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_{N-z})]\,q(R_2, \varphi_j, \tau_z) - l\sum_{z=1}^{N}\sum_{j=1}^{n}[\Phi_1^{II\,1}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_{N-z+1})$$
$$-\Phi_1^{II\,1}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_{N-z})]\,\eta(\varphi_j, \tau_z). \tag{25}$$

The expression (25) represents a linear system of algebraic equations with a given right-hand side at the time instant under consideration. Since for any time instant the right-hand side assumes different values and the left-hand side remains unchanged it is therefore expedient to find the matrix $A_{II\,2}^{-1}$, the inverse of the square matrix consisting of the coefficients $\Phi_1^{II\,2}(r_{k\Phi}, \Theta_i, \varphi_i, \tau_1)$ of the system (25). The sought solution is then found in the form

$$q(R_2, \Theta_i, \tau_N) = A_{\mathrm{II}\,2}^{-1}\{l\vartheta(r_{k\Phi}, \Theta_i, \tau_N) - \sum_{j=1}^{n} \Phi_{p1}^{\mathrm{II}}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_N)\, q_{\mathrm{st}}(R_2, \varphi_j)$$

$$-\sum_{z=1}^{N-1}\sum_{j=1}^{n}[\Phi_1^{\mathrm{II}\,2}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_{N-z+1}) - \Phi_1^{\mathrm{II}\,2}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_{N-z})]\, q(R_2, \varphi_j, \tau_z)$$

$$-l\sum_{z=1}^{N}\sum_{j=1}^{n}[\Phi_1^{\mathrm{II}\,1}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_{N-z+1}) - \Phi_1^{\mathrm{II}\,1}(r_{k\Phi}, \Theta_i, \varphi_j, \tau_{N-z})]\,\eta(\varphi_j, \tau_z)\}. \quad (26)$$

Equations (24) and (26) can be solved iteratively starting from the instant $\tau = 0$.

The solutions obtained by employing this procedure are identical with the results obtained by using an electrical model ([2] and [4]) and have an error up to 1.5%.

NOTATION

r	is the current radius of cylinder;
R_1	is the inner radius of cylinder;
R_2	is the outer radius of cylinder;
Θ and φ	are the central angles between 0 and 2π;
t	is the temperature, °C;
t_0	is the temperature of the inner surface of the cylinder for the steady-state problem, °C;
$\vartheta = t - t_0$, °C;	
q	is the density of the heat flux, W/m^2;
τ	is the time;
U	is the potential, %;
I	is the current, A.

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